

PERRON'S SOLUTIONS FOR TWO-PHASE FREE BOUNDARY PROBLEMS WITH DISTRIBUTED SOURCES

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ABSTRACT. We use Perron method to construct a weak solution to a two-phase free boundary problem with right-hand-side. We thus extend the results in [C3] for the homogeneous case.

1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in \mathbb{R}^n and let $A = A(x)$ be a symmetric matrix with Hölder continuous coefficients in Ω , which is uniformly elliptic, i.e.

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^n$$

for some $0 < \lambda \leq \Lambda$. Denote by

$$\mathcal{L} := \operatorname{div}(A(x)\nabla \cdot).$$

Let $f_1, f_2 \in L^\infty(\Omega)$. We consider the following two-phase inhomogeneous free boundary problem (f.b.p. in the sequel)

$$(1.1) \quad \begin{cases} \mathcal{L}u = f_1, & \text{in } \Omega^+(u) = \{u > 0\} \\ \mathcal{L}u = f_2 \chi_{\{u < 0\}} & \text{in } \Omega^-(u) = \{u \leq 0\}^\circ \\ u_\nu^+ = G(u_\nu^-, x, \nu) & \text{on } F(u) = \partial\{u > 0\} \cap \Omega. \end{cases}$$

Here $\nu = \nu(x)$ denotes the unit normal to $F(u)$ at x , pointing towards $\Omega^+(u)$. The function $G(\beta, x, \nu)$ is strictly increasing in β , Lipschitz continuous in all its arguments and $G(0) := \inf_{x \in \Omega, |\nu|=1} G(0, x, \nu) > 0$. Conormal derivatives $\nabla u^\pm \cdot \nu$ can be equally considered instead of normal derivatives.

Problems of this kind arise in several contexts, see [DFSs1] for a list.

In this paper, our main purpose is to construct a weak solution assuming given boundary data, via Perron method, extending the results of the seminal paper [C3] in the homogeneous case. Before stating our main result, we give the definition of weak solution of problem (1.1).

Given a continuous function v on Ω , we say that a point $x_0 \in F(v)$ is regular from the right (resp. left) if there is a ball $B \subset \Omega^+(v)$ (resp. $B \subset \Omega^-(v)$), such that $\overline{B} \cap F(v) = \{x_0\}$. In what follows, $\nu = \nu(x_0)$ represents the unit normal to ∂B at x_0 pointing toward $\Omega^+(v)$.

Definition 1.1. We say that $u \in C(\Omega)$ is a weak solution of the f.b.p. (1.1) if:

- a) $\mathcal{L}u = f_1$ in $\Omega^+(u)$ and $\mathcal{L}u = f_2 \chi_{\{u < 0\}}$ in $\Omega^-(u)$, in the usual weak sense;

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b) u satisfies the free boundary condition in (1.1) in the following sense:

(i) If $x_0 \in F(u)$ is regular from the right with touching ball B then

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \quad \text{in } B, \text{ with } \alpha \geq 0$$

and

$$u^-(x) \leq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \beta \geq 0$$

with equality along every non-tangential domain, and

$$\alpha \leq G(\beta, x_0, \nu(x_0)).$$

(ii) If $x_0 \in F(u)$ is regular from the left with touching ball B , then

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \quad \text{in } B, \text{ with } \beta \geq 0$$

$$u^+(x) \leq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \quad \text{in } B^c, \text{ with } \alpha \geq 0$$

with equality along every non-tangential domain, and

$$\alpha \geq G(\beta, x_0, \nu(x_0)).$$

Note that (i) (resp. (ii)) expresses a supersolution (resp. subsolution) condition at points regular from the right (resp. left). While this definition slightly differs from the one in [C3], it is indeed equivalent to it (see ([CS])).

Our solution is constructed as the infimum over an admissible class of supersolutions \mathcal{F} .

Definition 1.2. A function $w \in \mathcal{F}$ if $w \in C(\overline{\Omega})$ and if

(a) w is a weak solution to

$$\mathcal{L}w \leq f_1 \quad \text{in } \Omega^+(w) \quad \text{and} \quad \mathcal{L}w \leq f_2 \chi_{\{w < 0\}} \quad \text{in } \Omega^-(w).$$

(b) If $x_0 \in F(u)$ is regular from the left, then near x_0 ,

$$w^+ \leq \alpha \langle x - x_0, \nu(x_0) \rangle^+ + o(|x - x_0|), \quad \alpha \geq 0,$$

$$w^- \geq \beta \langle x - x_0, \nu(x_0) \rangle^- + o(|x - x_0|), \quad \beta \geq 0,$$

with

$$\alpha < G(\beta, x_0, \nu(x_0)).$$

(c) If $x_0 \in F(w)$ is not regular from the left, then near x_0 ,

$$w(x) = o(|x - x_0|).$$

We also need to introduce a minorant subsolution. We say that a locally Lipschitz function \underline{u} , defined in Ω , is a *minorant* if:

a) \underline{u} is a weak solution to

$$\mathcal{L}\underline{u} \geq f_1 \quad \text{in } \Omega^+(\underline{u}) \quad \text{and} \quad \mathcal{L}\underline{u} \geq f_2 \chi_{\{\underline{u} < 0\}} \quad \text{in } \Omega^-(\underline{u}).$$

b) Every $x_0 \in F(\underline{u})$ is regular from the right and near x_0 ,

$$\underline{u}^- \leq \beta \langle x - x_0, \nu(x_0) \rangle^- + o(|x - x_0|),$$

$$\underline{u}^+ \geq \alpha \langle x - x_0, \nu(x_0) \rangle^+ + o(|x - x_0|),$$

with

$$\alpha > G(\beta, x_0, \nu(x_0)).$$

We are now ready to state our main result.

Theorem 1.3. *Let ϕ be a continuous function on $\partial\Omega$ and \underline{u} be a minorant of our free boundary problem, with boundary data ϕ . Then*

$$u = \inf\{w : w \in \mathcal{F}, w \geq \underline{u} \text{ in } \overline{\Omega}\}$$

is a solution to (1.1) such that $u = \phi$ on $\partial\Omega$, as long as the set on the right is non-empty.

Concerning the regularity of the free boundary, we prove the following standard result.

Theorem 1.4. *The free boundary $F(u)$ has finite $(n-1)$ -dimensional Hausdorff measure. More precisely, there exists a universal constant $r_0 > 0$ such that for every $r < r_0$, for every $x_0 \in F(u)$,*

$$\mathcal{H}^{n-1}(F(u) \cap B_r(x_0)) \leq r^{n-1}.$$

Moreover, denoting with $F^(u)$ the reduced free boundary*

$$\mathcal{H}^{n-1}(F(u) \setminus F^*(u)) = 0.$$

In a forthcoming paper we shall adress further regularity properties of the free boundary. In particular, compactness properties of the minimal solutions constructed in Theorem 1.3 and the flatness result in [DFS4] will imply the following corollary, new even in the homogeneous case.

Theorem 1.5. *$F(u)$ is a $C^{1,\gamma}$ surface in a neighborhood of H^{n-1} a.e. point $x_0 \in F(u)$.*

The paper follows the main guidelines of [C3], although the presence of a distributed source requires to face new situations and requires new delicate arguments, especially in Sections 4 and 5. The organization is as follows. In Section 2 we prove some preliminary lemmas frequently used throughout the paper. In Section 3 we prove that u^+ is Lipschitz continuous. Then in Section 4 we show that u is Lipschitz continuous and it satisfies the equation in both $\Omega^+(u)$ and $\Omega^-(u)$. Linear growth near the free boundary and the non-degeneracy of u^+ are proved in Section 5. The following section, Section 6, is devoted to the proof that u satisfies the free boundary condition in the supersolution sense (part b(i). in Definition 1.1). Finally in Section 7 we prove that u satisfies the free boundary condition in the subsolution sense (part b(ii). in Definition 1.1) and hence it is a weak solution to our problem. We conclude our paper with the regularity result in Theorem 1.4 in Section 8.

Throughout the paper, constants depending possibly only on $[A]_{C^{0,\gamma}}, \|f_1\|_\infty, \|f_2\|_\infty, \lambda, \Lambda, G(0), n$ are called universal. Whenever a constant depends on other parameters, that dependance will be explicitly noted. Finally, for standard regularity theory for weak solution to divergence form equations, we refer the reader to [GT].

2. PRELIMINARIES

In this section we introduce some notation and prove some useful lemmas which will be used several times in the paper.

Notation. As usual, $B_r(x_0)$ denotes the ball in \mathbb{R}^n of radius r and center x_0 . When $x_0 = 0$ we omit the dependence on x_0 . Also, throughout the paper we will

use the following notation for rescalings of size r around x_0 :

$$(2.1) \quad \begin{aligned} u_r(x) &:= \frac{u(x_0 + rx)}{r}, \quad x \in \Omega_r := \frac{\{x - x_0 : x \in \Omega\}}{r}, \\ A_r(x) &= A(x_0 + rx), \quad f^r(x) = f(x_0 + rx), \quad \mathcal{L}_r = \operatorname{div}(A_r(x) \nabla \cdot) \\ G_r(\alpha, x, \nu) &= G(\alpha, x_0 + rx, \nu(x_0 + rx)). \end{aligned}$$

Finally, we denote

$$G(0) := \inf_{x \in \Omega, |\nu|=1} G(0, x, \nu) > 0.$$

Lemma 2.1. *Let $v \in C(\overline{B_r(x_0)})$, $v \geq 0$ for $r \leq 1$. Assume that*

$$\mathcal{L}v = f \in L^\infty \quad \text{in } B_r(x_0)$$

and

$$v(y_0) = 0, \quad y_0 \in \partial B_r(x_0).$$

Denote by ν the inner unit normal to $\partial B_r(x_0)$ at y_0 . Then,

$$v(x) \geq \alpha \langle x - y_0, \nu \rangle^+ + o(|x - y_0|)$$

with

$$\alpha \geq \bar{c} \frac{v(x_0)}{r} - \bar{C} r \|f\|_\infty$$

and $\bar{c}, \bar{C} > 0$ depending on $[A]_{0,\gamma}, \lambda, \Lambda, n$.

Proof. Let

$$v_r(x) = \frac{v(x_0 + rx)}{r}, \quad x \in B_1.$$

Then,

$$v_r \geq 0 \quad \text{in } B_1, \quad v_r(y_r) = 0, \quad y_r \in \partial B_1$$

and

$$\mathcal{L}_r v_r = r f^r \quad \text{in } B_1.$$

Notice that $A_r(x) = A(x_0 + rx)$ has the same ellipticity constants as A and its $C^{0,\gamma}$ norm is controlled by the $C^{0,\gamma}$ norm of A .

Call $\|f\|_\infty = M$. By Harnack inequality,

$$\inf_{B_{1/2}} v_r \geq c(v_r(0) - rM).$$

Now, denote with ξ, η the solutions to the the following problems:

$$\begin{aligned} \mathcal{L}_r \xi &= -1 \quad \text{in } B_1 \setminus \overline{B_{1/2}} \\ \xi &= 0 \quad \text{on } \partial B_{1/2}, \quad \xi = 0 \quad \text{on } \partial B_1, \\ \mathcal{L}_r \eta &= 0 \quad \text{in } B_1 \setminus \overline{B_{1/2}} \\ \eta &= 1 \quad \text{on } \partial B_{1/2}, \quad \eta = 0 \quad \text{on } \partial B_1. \end{aligned}$$

Call

$$c_1 = \xi_\nu|_{\partial B_1} > 0 \quad c_2 = \eta_\nu|_{\partial B_1} > 0,$$

with ν the inner unit normal to ∂B_1 . Notice that c_1 depends only on $[A]_{C^{0,\gamma}}, \lambda, \Lambda, n$.

Define,

$$\phi := c(v_r(0) - rM)\eta + rM\xi \quad \text{in } B_1 \setminus \overline{B_{1/2}}.$$

Then,

$$\mathcal{L}_r \phi = -rM \geq rf^r \quad \text{in } B_1 \setminus \overline{B_{1/2}},$$

and

$$\phi \leq v_r \quad \text{on } \partial B_1 \cup \partial B_{1/2}.$$

Thus,

$$\phi \leq v_r \quad \text{in } B_1 \setminus \overline{B_{1/2}},$$

and hence

$$v_r(x) \geq (c(v_r(0) - rM)c_1 + rMc_2)\langle x - y_r, \nu \rangle^+ + o(|x - y_r|),$$

which gives the desired result. \square

Next we prove the following asymptotic developments lemmas.

Lemma 2.2. *Let Ω be an open set, $0 \in \partial\Omega$. Assume that $B_\rho(-\rho e^1) \subset R^n \setminus \overline{\Omega}$. Let u be a nonnegative Lipschitz function in $B_1 \cap \overline{\Omega}$, satisfying $\mathcal{L}u = f$ in $B_1 \cap \Omega$ and $u = 0$ on $\partial\Omega \cap B_1$.*

Then there exists $\alpha \geq 0$ such that

$$u(x) = \alpha x_1 + o(|x|) \quad \text{as } x \rightarrow 0, x \in \overline{\Omega} \cap B_1.$$

In particular, if $\alpha > 0$, then along $\partial\Omega$,

$$x_1 = o(|x|) \quad \text{as } x \rightarrow 0, x \in \partial\Omega \cap B_1$$

that is $\partial\Omega$ is tangent to the hyperplane $x_1 = 0$.

Proof. We may assume that $\rho < 1/3$. We change variables by setting

$$y = T(x) = \frac{e^1}{\rho} - \frac{x + \rho e^1}{|x + \rho e^1|^2}$$

and define $v(y) = u(T^{-1}(y))$. Then $T(0) = 0$, and the exterior of the ball $B_\rho(-\rho e^1)$ is mapped onto $B_{1/\rho}(e^1/\rho) \setminus \{e^1/\rho\}$. Thus $\Omega' = T(\Omega) \subset B_{1/\rho}(e^1/\rho)$ and $\Omega' \cap B_2 \subset B_2^+ = \{y \in B_2 : y_1 > 0\}$.

Note also that,

$$(2.2) \quad y_1 = \left(\frac{2}{\rho^2} - 1\right) x_1 + o(|x|).$$

Moreover, v is Lipschitz in $\overline{\Omega'} \cap B_2$, $v = 0$ on $\partial\Omega' \cap B_2$ and

$$\mathcal{L}'v = \operatorname{div}(A'(y) \nabla v) = f'(y) \equiv f(T^{-1}(y)) \cdot |\det J| \quad \text{in } \Omega' \cap B_2$$

where $A' = JAJ^\top \cdot |\det J|$, J being the Jacobian of T^{-1} . Note that if A is symmetric then A' is symmetric and

$$c(\rho, \lambda) I \leq A'(y) \leq C(\rho, \Lambda) I \quad \text{in } \overline{\Omega'} \cap B_2.$$

Extend v by zero in B_1 outside Ω' . Then (still calling v the extended function), $\mathcal{L}'v \geq -\|f'\|_\infty$ in B_2^+ (in a weak sense). We also have $v(y) \leq Cy_1$ in $B_{3/2}^+$ (compare with the solution of $\mathcal{L}'z = -\|f'\|_\infty$, $z = v$ on ∂B_2^+).

Now, let $w = w(x)$ be the \mathcal{L}' -harmonic measure in B_2^+ of $S_2^+ = \partial B_2 \cap \{y_1 > 0\}$. Then, by Hopf principle and standard regularity theory,

$$(2.3) \quad y_1 c_1 \leq w(y) \leq c_2 y_1 \quad \text{in } \overline{B_1^+}$$

with c_1, c_2 positive and universal, and, for some universal $\gamma > 0$,

$$(2.4) \quad w(y) = \gamma y_1 + o(|y|) \quad \text{as } y \rightarrow 0, y \in B_1^+.$$

Let now for $k \geq 1$, integer,

$$m_k = \inf \left\{ m : v(y) \leq mw(y) \quad \text{for every } y \in \overline{\Omega}' \cap B_{1/k} \right\}.$$

Then $\{m_k\}$ is non increasing and $m_k \rightarrow m_\infty \geq 0$. Moreover

$$(2.5) \quad v(y) \leq m_\infty w(y) + o(|y|) \quad \text{as } y \rightarrow 0, y \in \overline{\Omega}' \cap B_1.$$

We claim that equality holds in (2.5). If not, there exist $\delta > 0$ and a sequence $\{y_j\} \in \Omega' \cap B_1$ such that $r_j = |y_j| \rightarrow 0$ and

$$v(y_j) \leq m_\infty w(y_j) - \delta r_j.$$

Since both v and w are Lipschitz, we can write

$$(2.6) \quad W(y) \equiv m_\infty w(y) - v(y) \geq \delta r_j / 2 \quad \text{on } B_{cr_j}(y_j) \cap S_{r_j}^+$$

with c depending on m_∞ and the Lipschitz constants of v and w .

On the other hand, (2.5) implies that

$$(2.7) \quad W(y) \geq -\sigma_j r_j \quad \text{on } S_{r_j}^+$$

with $\sigma_j \rightarrow 0$. Rescale by setting

$$W_j(y) = m_\infty \frac{w(r_j y)}{r_j} - \frac{v(r_j y)}{r_j} = \frac{W(r_j y)}{r_j} \quad y \in B_1^+.$$

Note that (2.3) still holds for $w(r_j y)/r_j$. Then $W_j(y) = 0$ on $y_1 = 0$, $W_j(y) \geq -\sigma_j$ on S_1^+ , $W_j(y) \geq \delta/2$ on $B_c(y_j/r_j) \cap S_1^+$. Moreover, setting $\mathcal{L}'_j = \operatorname{div}(A'(r_j y) \nabla)$,

$$\mathcal{L}'_j W_j \leq r_j \|f'\|_\infty \quad \text{in } B_1^+.$$

By Hopf principle and standard comparison, in $B_{1/2}^+$ we can write

$$W_j(y) \geq (-c_3 \sigma_j - c_4 r_j \|f'\|_\infty + c_5 \delta/2) y_1$$

with c_3, c_5 universal and c_4 depending on the Lipschitz constant of v . For j large enough, we get, say

$$m_\infty w_j(y) - v_j(y) \geq \frac{\delta}{100} y_1.$$

Rescaling back and using (2.3), we get a contradiction to the definition of m_∞ . Thus we have equality in (2.5) and taking into account (2.4), we get

$$v(x) = \gamma m_\infty y_1 + o(|y|) \quad \text{as } x \rightarrow 0, x \in \overline{\Omega}' \cap B_1.$$

Going back to the original variables, from (2.2), we get

$$u(x) = \alpha x_1 + o(|x|) \quad \text{as } x \rightarrow 0, x \in \overline{\Omega} \cap B_1$$

with $\alpha = \left(\frac{2}{\rho^2} - 1\right) \gamma m_\infty$. □

Lemma 2.3. *Let Ω be an open set, $0 \in \partial\Omega$. Assume that*

$$(2.8) \quad B_\rho(\rho e^1) \subset \Omega$$

Let u be a nonnegative Lipschitz function in $B_2 \cap \overline{\Omega}$, satisfying $\mathcal{L}u = f$ in $B_2 \cap \Omega$ and $u = 0$ on $\partial\Omega \cap B_2$.

Then there exists $\alpha \geq 0$ such that

$$u(x) = \alpha x_1 + o(|x|) \quad \text{as } x \rightarrow 0, x \in B_\rho(\rho e^1).$$

Proof. After a smooth change of variables (e.g. flattening the surface ball) which leaves both the origin and the normal direction at 0 fixed, we may replace (2.8) by

$$B_2^+ \subset \Omega$$

always with $0 \in \partial\Omega$. We keep the same notation u and \mathcal{L} for the transformed u and the new operator, which is uniformly elliptic with ellipticity constant of the same order of λ, Λ . As in Lemma 2.2, let $w = w(x)$ be the \mathcal{L} -harmonic measure in B_2^+ of $S_2^+ = \partial B_2 \cap \{y_1 > 0\}$. For $k \geq 1$, integer, define

$$\alpha_k = \sup \left\{ \alpha : u(x) \geq \beta w(x) \quad \text{for every } x \in B_{1/k}^+ \right\}.$$

Then $\{\alpha_k\}$ is nondecreasing and $\alpha_k \rightarrow \alpha \geq 0$. Moreover

$$(2.9) \quad u(x) \geq \alpha w(x) + o(|x|) \quad \text{as } x \rightarrow 0, x \in B_1^+.$$

We claim that equality holds in (2.5). If not, there exist $\delta > 0$ and a sequence $\{x_j\} \in B_1^+$ such that $r_j = |x_j| \rightarrow 0$ and

$$u(x_j) - \alpha w(x_j) \geq \delta r_j.$$

By Lipschitz continuity, we can write

$$(2.10) \quad U(x) \equiv u(x) - \alpha w(x) \geq \delta r_j / 2 \quad \text{on } B_{cr_j}(x_j) \cap S_{r_j}^+$$

with c depending on α and the Lipschitz constants of u and w .

On the other hand, (2.5) implies that

$$(2.11) \quad U(x) \geq -\sigma_j r_j \quad \text{on } S_{r_j}^+$$

with $\sigma_j \rightarrow 0$. Rescale by setting

$$U_j(x) = \frac{u(r_j x)}{r_j} - \alpha \frac{w(r_j x)}{r_j} = \frac{U(r_j x)}{r_j} \quad x \in B_1^+.$$

Then $U_j(0) = 0$, $U_j(x) \geq -\sigma_j$ on S_1^+ , $U_j(x) \geq \delta/2$ on $B_c(x_j/r_j) \cap S_1^+$. Moreover, setting $\mathcal{L}_j = \text{div}(A(r_j x) \nabla)$,

$$\mathcal{L}_j U_j \leq r_j \|f\|_\infty \quad \text{in } B_1^+.$$

By Hopf principle and standard arguments, in $B_{1/2}^+$ we can write, for j large

$$U_j(x) \geq (-c\sigma_j - c_0 r_j \|f\|_\infty + C\delta/2)x_1 \geq \frac{\delta}{100}x_1.$$

Rescaling back and using (2.3), we get a contradiction to the definition of α . \square

Remark 2.4. We remark that the expansions in the lemmas above remain valid if we replace the assumption that u is Lipschitz with the existence of a touching ball at 0 both from the right and from the left.

3. LIPSCHITZ REGULARITY OF u^+ .

In this section we prove that u^+ is Lipschitz continuous. In order to follow the strategy developed in [C3], we need the following ‘‘almost-monotonicity’’ formula, see [MP].

Proposition 3.1. *Let $u_i, i = 1, 2$ be continuous functions in the unit ball B_1 that satisfy*

$$u_i \geq 0, \quad \mathcal{L}u_i \geq -1, \quad u_1 \cdot u_2 = 0 \quad \text{in } B_1.$$

Then there exist universal constants C_0 and r_0 , such that the functional

$$\Phi(r) := r^{-4} \int_{B_r} \frac{|\nabla u_1|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla u_2|^2}{|x|^{n-2}} dx$$

satisfies

$$\Phi(r) \leq C_0(1 + \|u_1\|_{L^2(B_1)}^2 + \|u_2\|_{L^2(B_1)}^2)^2$$

for $0 < r < r_0$.

Remark 3.2. We remark that, by Fubini's theorem

$$\int_{B_R} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx = R^{2-n} \int_{B_R} |\nabla u_i|^2 dx + (n-2)R^{-2} \int_0^r \left(\int_{B_r} |\nabla u_i|^2 \right) r^{1-n} dr.$$

Remark 3.3. We remark that if v satisfies $\mathcal{L}v \geq -M$ say in $B_1^+(v)$, then $\mathcal{L}v^+ \geq -M$ in B . This follows by standard arguments. Indeed, if $\psi_\varepsilon(t)$ is a convex increasing function such that $\psi_\varepsilon(t) = 0$ for $t \leq \varepsilon$, then it is easy to see that

$$\mathcal{L}\psi_\varepsilon(v) \geq \psi'_\varepsilon(u_\varepsilon)\mathcal{L}v \geq -M \quad \text{in } B_1.$$

The desired result follows by approximating t^+ with a sequence of ψ_ε .

The next lemma is the first step towards proving that u^+ is Lipschitz. The standard technique of harmonic replacement cannot be applied in our case, as we are not imposing any sign condition on the right-hand-side f_1 . We bypass this difficulty solving an obstacle-type problem.

Lemma 3.4. *Let $w \in \mathcal{F}$, then there exists $\tilde{w} \in \mathcal{F}$ such that*

- (i) $\mathcal{L}\tilde{w} = f_1$ in $\Omega^+(\tilde{w})$,
- (ii) $\tilde{w}^+ \leq w$, $\tilde{w}^- = w$
- (iii) $\tilde{w} \geq \underline{u}$, $\tilde{w} = \phi$ on $\partial\Omega$.

Proof. Let $w \in \mathcal{F}$. For notational simplicity call $\Omega^+ = \Omega^+(w)$ and set

$$\mathcal{S} = \{v \in C(\bar{\Omega}^+) : \mathcal{L}v \geq f_1 \chi_{\{v>0\}} \text{ in } \Omega^+, v \geq 0 \text{ in } \Omega^+, v = w \text{ on } \partial\Omega^+\}.$$

Notice that $\mathcal{S} \neq \emptyset$ since $\underline{u}^+ \in \mathcal{S}$. Also, if $v \in \mathcal{S}$ then $v \leq w$ in Ω^+ . Define,

$$\tilde{w} := \sup \mathcal{S}.$$

Then $\tilde{w} \leq w$ and solves the obstacle problem (see [KS])

$$\begin{cases} \mathcal{L}\tilde{w} = f_1 & \text{in } \{\tilde{w} > 0\}, \quad \tilde{w} \geq 0 \quad \text{in } \Omega^+ \\ \tilde{w} = w & \text{on } \partial\Omega^+. \end{cases}$$

By the regularity theory for the obstacle problem we conclude that \tilde{w} is locally $C^{1,\gamma}$ in Ω^+ (see [T]).

Extend \tilde{w} to $\bar{\Omega}$ by setting

$$\tilde{w} = w \quad \text{in } \bar{\Omega} \cap \{w \leq 0\}.$$

Hence by definition, $\tilde{w} \geq \underline{u}$ on Ω and $\tilde{w} = g$ on $\partial\Omega$.

To conclude that $\tilde{w} \in \mathcal{F}$ we only need to show that \tilde{w} satisfies the free boundary condition in the sense of Definition 1.2.

Let $x_0 \in F(\tilde{w})$, then either $x_0 \in F(w)$ or $x_0 \in \Omega^+ \cap \partial\{\tilde{w} = 0\}$. In the latter case, by the $C^{1,\gamma}$ regularity of \tilde{w} we immediately obtain that the free boundary condition is satisfied, possibly with $\alpha = \beta = 0$ (recall $G(0) > 0$.) If $x_0 \in F(w)$ then the conclusion follows immediately from the fact that $\tilde{w} \leq w$ in Ω^+ and $\tilde{w} = w$ otherwise. \square

The following result is a consequence of the weak monotonicity formula.

Theorem 3.5. *Let $w \in \mathcal{F}$ and $\mathcal{L}w = f_1$ in $\Omega^+(w)$. Then, w^+ is locally Lipschitz in Ω . Moreover, denoting by*

$$G^{-1}(\alpha) = \inf_{x, \nu} G^{-1}(\alpha, x, \nu),$$

for any $D \subset\subset \Omega$, w^+ is Lipschitz in D with Lipschitz constant L_D satisfying

$$(3.1) \quad L_D G^{-1}(L_D) \leq C(1 + \|w^+\|_{L^2(D)}^2 + \|w^-\|_{L^2(D)}^2)$$

and C depending on D .

Proof. Let $x_0 \in F(w)$ be a regular point from the left where w has the asymptotic development

$$\begin{aligned} w^+ &= \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \alpha > 0 \\ w^- &\geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \beta \geq 0, \end{aligned}$$

with

$$\alpha < G(\beta, x_0, \nu).$$

Let us show that

$$\alpha G^{-1}(\alpha) \leq C(1 + \|w^+\|_{L^2}^2 + \|w^-\|_{L^2}^2)^2,$$

with C depending on $\text{dist}(x_0, \partial\Omega)$. We will use Proposition 3.1. Notice that in view of Remark 3.3, the conclusion of Proposition 3.1 holds for $u_1 = w^+, u_2 = w^-$.

If $G^{-1}(\alpha) = 0$, then there is nothing to prove. Thus, let $G^{-1}(\alpha) > 0$ and let us prove that

$$\alpha^2 \beta^2 \leq C(1 + \|w^+\|_{L^2}^2 + \|w^-\|_{L^2}^2)^2,$$

from which the desired inequality will follow.

For convenience, use coordinates $x = (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and assume that $x_0 = (0, 0), \nu = (0, 1)$. Following [C3] pg. 587, one can estimate that as $s \rightarrow 0$,

$$\int_{B_s} |\nabla w^+|^2 dx \geq \int_{B_s \cap \{y > 0\}} (\alpha^2 - o(1)) dx$$

and

$$\int_{B_s} |\nabla w^-|^2 dx \geq \int_{B_s \cap \{y > 0\}} (\beta^2 - o(1)) dx.$$

Thus, by Remark 3.2, for all sufficiently small s , if Φ is the functional defined in Proposition 3.1

$$(3.2) \quad \Phi(s) \geq c_n s^{-4} \int_0^s (\alpha^2 - o(1)) r dr \int_0^s (\beta^2 - o(1)) r dr,$$

with $c_n = 16/\omega_n^2$ (ω_n the measure of the unit sphere.)

Hence, for all r small $\alpha^2 \beta^2 \leq C\Phi(r)$ which together with the conclusion of Proposition 3.1 gives the desired estimate.

Now, let $x_0 \in \Omega^+(w) \cap D$ and let

$$\text{dist}(x_0, F(w)) = |x_0 - y_0| = r < \frac{1}{2} \text{dist}(D, \partial\Omega),$$

say $r \leq 1$.

To prove the result it is sufficient to prove the existence of a positive constant M such that

$$\frac{w(x_0)}{r} \leq M.$$

Suppose

$$\frac{w(x_0)}{r} > M,$$

with M to be specified later. By Lemma 2.1, we have that

$$w(x) \geq \alpha_M \langle x - y_0, \nu \rangle^+ + o(|x - y_0|)$$

with

$$\alpha_M = \bar{c}M - \bar{C}r\|f_1\|_\infty.$$

For M large $\alpha_M > 0$ and y_0 is regular from the left. Then according to Lemma 2.4

$$w(x) = \alpha \langle x - y_0, \nu \rangle^+ + o(|x - y_0|)$$

and $\alpha \geq \alpha_M$.

Hence we can apply the previous estimate and conclude that

$$\alpha_M G^{-1}(\alpha_M) \leq C(1 + \|w^+\|_{L^2}^2 + \|w^-\|_{L^2}^2)^2.$$

The contradiction follows for M large, since $\alpha_M G^{-1}(\alpha_M) \rightarrow \infty$ as $M \rightarrow \infty$. \square

As a corollary of the two results above we obtain the following.

Corollary 3.6. *u^+ is locally Lipschitz and it satisfies*

$$\mathcal{L}u = f_1 \quad \text{in } \Omega^+(u).$$

4. THE FUNCTION u IS LIPSCHITZ

In this section we show that u^- is Lipschitz. First, we prove the following standard lemma.

Lemma 4.1. *If $w_1, w_2 \in \mathcal{F}$, then*

$$w^* = \min\{w_1, w_2\}.$$

Proof. The fact that $\mathcal{L}w^* = f_1$ in $\Omega^+(w^*)$ and $\mathcal{L}w^* = f_2\chi_{\{w^* < 0\}}$ follows from standard arguments. To prove that w^* solves the free boundary condition in the sense of Definition 1.2 it suffices to notice that $\Omega^+(w^*) = \Omega^+(w_1) \cap \Omega^+(w_2)$. Hence, any ball touching $F(w_1)$ or $F(w_2)$ from the left will also touch $F(w^*)$ from the left. Thus, if x_0 is not regular for $F(w^*)$, it cannot be regular for $F(w_1)$ and $F(w_2)$ either and near x_0 , $w^*(x) = w_1(x) = w_2(x) = o(|x - x_0|)$. If x_0 is regular, the asymptotic developments for w^* come from those of w_1 or w_2 (possibly with $\alpha = \beta = 0$ in case x_0 is not regular for either $F(w_1)$ or $F(w_2)$.) \square

The proof of the Lipschitz continuity of u^- is based on the following replacement technique. The harmonic replacement technique in [C3] does not work in this context. Instead, we need to perform a replacement with solutions to obstacle-type problems.

Precisely, let $w \in \mathcal{F}$ and let $w(x_0) < 0$. Let $B := B_R(x_0)$ be a ball around x_0 . Denote by

$$\Omega_1 := \Omega^+(w) \setminus \bar{B}.$$

Define

$$\mathcal{S}_1 = \{v : \mathcal{L}v \geq f_1 \chi_{\{v>0\}} \text{ in } \Omega_1, v \geq 0 \text{ in } \Omega_1, v = w \text{ on } \partial\Omega_1 \setminus \partial B, v = 0 \text{ on } \partial B\}.$$

Notice that since \underline{u} is locally Lipschitz and $\underline{u}(x_0) \leq w(x_0) < 0$, \underline{u} is strictly negative in B for R small. Hence $\underline{u}^+ \in \mathcal{S}_1$ and \mathcal{S}_1 is non empty. Let

$$w_1 = \sup \mathcal{S}_1.$$

Then w_1 solves the obstacle problem (see [KS, T])

$$(4.1) \quad \mathcal{L}w_1 = f_1 \quad \text{in } \{w_1 > 0\}, \quad w_1 \geq 0.$$

Analogously, define

$$\mathcal{S}_2 = \{v : \mathcal{L}v \geq -f_2 \chi_{\{v>0\}} \text{ in } B, v \geq 0 \text{ in } B, v = w^- \text{ on } \partial B\}.$$

Clearly $\mathcal{S}_2 \neq \emptyset$ since $w^- \in \mathcal{S}_2$. Let,

$$w_2 = \sup \mathcal{S}_2.$$

Again, w_2 solves the obstacle problem

$$(4.2) \quad \mathcal{L}w_2 = -f_2 \quad \text{in } \{w_2 > 0\}, \quad w_2 \geq 0.$$

We define the “double-replacement” \tilde{w} of w in B as follows

$$\tilde{w} = \begin{cases} w_1 & \text{in } \bar{\Omega}_1 \\ -w_2 & \text{in } \bar{B} \\ w & \text{otherwise.} \end{cases}$$

By construction $\tilde{w} \leq w$. Indeed in Ω_1 this follows by the maximum principle, while in B it follows from the fact that $w^- \in \mathcal{S}_2$.

We wish to prove the following lemma.

Lemma 4.2. *Let $w \in \mathcal{F}$, $w(x_0) = -h$. Then there exists ϵ (depending on $\text{dist}(x_0, \partial\Omega)$ and \underline{u}) such that*

- (i) *The double replacement \tilde{w} of w in $B_{\epsilon h}(x_0)$ belongs to \mathcal{F} and $\underline{u} \leq \tilde{w} \leq w$.*
- (ii) *$\mathcal{L}\tilde{w} = f_2$ and $\tilde{w} < 0$ in $B_{\epsilon h}(x_0)$ and*

$$|\nabla \tilde{w}| \leq \frac{C}{\epsilon} + \epsilon C \|f_2\|_\infty \quad \text{in } B_{\epsilon h/2}(x_0).$$

Proof. We already noticed that $\tilde{w} \leq w$. Now we observe that $\tilde{w} \geq \underline{u}$. As already remarked, since \underline{u} is locally Lipschitz and $\underline{u}(x_0) < -h$, \underline{u} is strictly negative in $B := B_{\epsilon h}(x_0)$ for ϵ small. Thus, $\underline{u}^+ \in \mathcal{S}_1$ and $w_1 \geq \underline{u}$. Also, by the maximum principle in $\{w_2 > 0\}$, it follows that $-w_2 \geq \underline{u}$ in B . Hence $\tilde{w} \geq \underline{u}$. We denote with $-m = \min_{\bar{\Omega}} \underline{u}$, $m > 0$. Notice that $h \leq m$.

Let us also observe that, for ϵ small, $w_2 > 0$ in $B_{\epsilon h}(x_0)$. Indeed if $\partial\{w_2 > 0\} \cap B_{\epsilon h}(x_0) \neq \emptyset$, then by the growth of the solution to the obstacle problem, we get

$$w_2(x_0) \leq C(\epsilon h)^2.$$

For ϵ small, this contradicts that $-w_2(x_0) \leq w(x_0) = -h$. In particular, it follows from (4.2) that

$$(4.3) \quad \mathcal{L}\tilde{w} = f_2 \quad \text{in } B_{\epsilon h}(x_0).$$

Now, the fact that \tilde{w} satisfies (a) in Definition 1.2 follows from (4.1)-(4.3) and standard arguments.

We need to verify that the free boundary condition is satisfied in the sense of part (b) in Definition 1.2. Let $\bar{x} \in F(\tilde{w})$. Then three possibilities can occur. If $x_1 \in F(w)$, we use that $\tilde{w} \leq w$ and hence \tilde{w} has the correct asymptotic behavior whether x_1 is regular or not (recall $G(0, \cdot, \cdot) > 0$). If $x_1 \in \partial\{w_1 > 0\} \cap \Omega^+(w)$ then by the regularity of the solution to the obstacle problem we get again that \tilde{w} has the correct asymptotic behavior. Finally, we consider the case when $x_1 \in \partial B \cap \Omega^+(w)$. Since w^+ is locally Lipschitz, say with constant L in $B_{d_0/2}(x_0)$ we get that

$$\tilde{w} \leq w^+ \leq L\epsilon h \quad \text{in } B_{2\epsilon h}(x_0).$$

Let us rescale and using the notation in (2.1) call

$$\tilde{w}_\epsilon(x) = \frac{\tilde{w}(x_0 + \epsilon h x)}{\epsilon h}.$$

Then,

$$\tilde{w}_\epsilon \leq L \quad \text{in } B_2.$$

Let us call v_1, v_2 the solutions to the the following problems:

$$\begin{aligned} \mathcal{L}_\epsilon v_1 &= 0 \quad \text{in } B_2 \setminus \overline{B_1} \\ v_1 &= 0 \quad \text{on } \partial B_1, \quad v_1 = 1 \quad \text{on } \partial B_2, \\ \mathcal{L}_\epsilon v_2 &= -1 \quad \text{in } B_2 \setminus \overline{B_1} \\ v_2 &= 0 \quad \text{on } \partial B_1, \quad v_2 = 0 \quad \text{on } \partial B_2. \end{aligned}$$

Define,

$$v := Lv_1 + \epsilon h M v_2$$

with $M = \|f_1\|_\infty$. Then, applying the maximum principle in $(B_2 \setminus \overline{B_1}) \cap \Omega^+(\tilde{w}_\epsilon)$ we obtain that

$$\tilde{w}_\epsilon^+ \leq v \quad \text{in } B_2 \setminus \overline{B_1}.$$

Thus,

$$\tilde{w}_\epsilon^+ \leq \alpha \langle x - (x_1)_\epsilon, \nu((x_1)_\epsilon) \rangle^+ + o(|x - (x_1)_\epsilon|)$$

with

$$\alpha = (Lc_1 + \epsilon h M c_2), \quad c_1 = (v_1)_\nu|_{\partial B_1}, \quad c_2 = (v_2)_\nu|_{\partial B_1}$$

$$(x_1)_\epsilon = \frac{x_1 - x_0}{\epsilon h}$$

and $\nu(y)$ the normal to ∂B_1 at y pointing outside B_1 .

In terms of \tilde{w} this gives,

$$(4.4) \quad \tilde{w}^+ \leq \alpha \langle x - x_1, \nu \rangle^+ + o(|x - x_1|), \quad \alpha \leq \bar{L},$$

and ν the exterior unit normal to $\partial B_{\epsilon h}(x_0)$ at x_1 .

On the other hand, by Lemma 2.1 applied to $-(w_2)_\epsilon$ we have that

$$\tilde{w}_\epsilon^- = -(w_2)_\epsilon \geq \beta \langle x - (x_1)_\epsilon, \nu((x_1)_\epsilon) \rangle^- + o(|x - (x_1)_\epsilon|),$$

with $(\bar{M} = \|f_2\|_\infty)$

$$\beta \geq \frac{\bar{c}}{\epsilon} - \bar{C}\epsilon h \bar{M}.$$

In terms of \tilde{w} and for ε small, this implies that

$$(4.5) \quad \tilde{w}^- \geq \beta \langle x - x_1, \nu \rangle^- + o(|x - x_1|), \quad \beta \geq \frac{\bar{c}}{2\varepsilon}.$$

In view of (4.4)-(4.5), the free boundary condition is satisfied if we choose ε small enough so that

$$\bar{L} < \inf_{x, \nu} G\left(\frac{\bar{c}}{\varepsilon}, \cdot, \cdot\right).$$

Finally, the estimate in (ii) follows from standard Schauder estimates and Harnack inequality. \square

We obtain the following immediate corollary.

Corollary 4.3. *Let x_0 be a point where $u(x_0) = -h < 0$. Then, there exists a non-increasing sequence $\{\tilde{w}_j\} \subset \mathcal{F}$, $\tilde{w} \geq \underline{u}$, and $\varepsilon > 0$, depending on $d_0 = \text{dist}(x_0, \partial\Omega)$, such that the following hold:*

- (i) $\tilde{w}_k(x_0) \searrow u(x_0)$;
- (ii) $\mathcal{L}\tilde{w}_k = f_2$ and $\tilde{w}_k < 0$ in $B_{\varepsilon h}(x_0)$;
- (iii) For each k , \tilde{w}_k is Lipschitz in $B_{\varepsilon h/2}(x_0)$ with Lipschitz constant L_0 depending on d_0 .

Finally, we can finish the proof that u satisfies part a) in Definition 1.1.

Corollary 4.4. *u is locally Lipschitz in Ω , continuous in $\bar{\Omega}$, $u = g$ on $\partial\Omega$. Moreover u solves*

$$\mathcal{L}u = f_2 \chi_{\{u < 0\}}, \quad \text{in } \Omega^-(u).$$

Proof. Let $u(x_0) = -h < 0$ and let $\{\tilde{w}_k\}$ be as in the lemma above. We want to prove that $\tilde{w}_k \searrow u$ uniformly, say on $B_{h\varepsilon/4}$. Indeed suppose by contradiction that there exists $x_1 \in B_{\varepsilon h/4}(x_0)$ where $\tilde{w}(x_1) = \lim_{j \rightarrow \infty} \tilde{w}_j(x_1) > u(x_1)$. Then consider a new sequence $\{v_j\}_{j \in \mathbb{N}}$ converging to u at x_1 , and define $\{\tilde{u}_k\}$ as a replacement of $\{\min\{\tilde{v}_k, \tilde{w}_k\}\}_{k \in \mathbb{N}}$ in $B_{\varepsilon h/2}(x_0)$. Then $\lim_{k \rightarrow \infty} \tilde{u}_k = \tilde{u}$ decreasing with $\tilde{u} \leq \tilde{w}$ in $B_{\varepsilon h/2}(x_0)$, $\tilde{u}(x_0) = \tilde{w}(x_0)$ and $\tilde{u}(x_1) < \tilde{w}(x_1)$. Moreover in $B_{\varepsilon h/4}(x_0)$

$$\mathcal{L}(\tilde{w} - \tilde{u}) = 0,$$

$\tilde{w} - \tilde{u} \geq 0$ and $(\tilde{w} - \tilde{u})(x_0) = 0$ and by maximum principle it follows that $\tilde{w} - \tilde{u} \equiv 0$ obtaining a contradiction with $(\tilde{w} - \tilde{u})(x_1) > 0$. As a consequence u satisfies

$$\mathcal{L}u = f_2 \quad \text{in } \{u < 0\}.$$

\square

Corollary 4.5. *If K is compactly contained in Ω , then u is uniform limit of a sequence of functions $\{w_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$ in K . If $K \Subset \Omega^-(u)$, $\{w_k\}_{k \in \mathbb{N}}$ may be taken non-positive in \bar{K} .*

Proof. The first part follows from the fact that $\{w^+ : w \in \mathcal{F}\}$ is equicontinuous in \bar{K} and from the previous replacement technique.

By compactness, it is enough to prove the second part for balls $B_\varepsilon(x_0) \Subset \Omega^-(u)$, with ε small enough. Let $w_k \searrow u$ uniformly in $\bar{B}_{2\varepsilon}(x_0) \Subset \Omega^-(u)$. Let us rescale by ε and use the notation in (2.1). Let η, ξ solve the following problems:

$$\begin{aligned} \mathcal{L}_\varepsilon \eta &= 0 \quad \text{in } B_2 \setminus \bar{B}_1 \\ \eta &= 0 \quad \text{on } \partial B_1, \quad \eta = 1 \quad \text{on } \partial B_2, \end{aligned}$$

$$\begin{aligned}\mathcal{L}_\varepsilon \xi &= -1 \quad \text{in } B_2 \setminus \overline{B_1} \\ \xi &= 0 \quad \text{on } \partial B_1, \quad \xi = 0 \quad \text{on } \partial B_2.\end{aligned}$$

Call

$$c_0 := \eta_\nu|_{\partial B_1} > 0, \quad c_1 := \xi_\nu|_{\partial B_1} > 0$$

with ν the unit normal to ∂B_1 pointing inward and let $c_2 > 0$ be such that

$$(4.6) \quad c_0 c_2 < \frac{G(0)}{2}.$$

Define, ($M = \|f_2\|_\infty$, say $M > 0$)

$$v := \varepsilon M \xi + c_2 \eta \quad \text{in } B_2 \setminus \overline{B_1}, \quad v \equiv 0 \quad \text{on } B_1.$$

Then,

$$\mathcal{L}_\varepsilon v = -\varepsilon M \leq \varepsilon f_2^\varepsilon \quad \text{in } B_2 \setminus \overline{B_1}.$$

Since $u_\varepsilon \leq 0$ in $\overline{B_2}$, for k sufficiently large $w_k \leq \varepsilon M/2$ in $\overline{B_2}$. Define

$$\bar{w}_k = \begin{cases} \min\{w_k^\varepsilon, v\} & \text{in } \overline{B_2}, \\ w_k^\varepsilon, & \text{otherwise.} \end{cases}$$

Then, in view of (4.6), as long as

$$\varepsilon < \frac{G(0)}{2Mc_1}$$

the function

$$\bar{w}_k(x) = \varepsilon \bar{w}_k^\varepsilon \left(\frac{x - x_0}{\varepsilon} \right)$$

satisfies

$$\bar{w}_k \in \mathcal{F}, \quad \bar{w}_k \leq 0 \quad \text{in } \overline{B_\varepsilon}(x_0)$$

and $\bar{w}_k \searrow u$ in $\overline{B_\varepsilon}(x_0)$, as desired. □

5. ON THE DEGENERACY OF u^+

In this section we prove that u^+ is not degenerate. As a consequence $F(w_k) \rightarrow F(u)$ locally in Hausdorff distance and $\chi_{\{w_k > 0\}} \rightarrow \chi_{\{u > 0\}}$ in $L^1_{loc}(\Omega)$.

First, we recall the following standard lemma.

Lemma 5.1. *Let u be a Lipschitz function in $\overline{\Omega} \cap B_1(0)$ satisfying $\mathcal{L}u = f$, vanishing on $\partial\Omega \cap B_1$ and $0 \in \partial\Omega$. Suppose that there exists a positive constant C such that for every $x \in B_{1/2} \cap \Omega$*

$$(5.1) \quad u(x) \geq c \text{dist}(x, \partial\Omega).$$

Then there exists a constant $C > 0$ such that

$$\sup_{B_r(0)} u \geq Cr,$$

for all $r \leq r_0$ universal.

Proof. Let $\text{dist}(x_0, \partial\Omega) = \epsilon$. Then by (5.1) and the Lipschitz continuity of u (say $L = \text{Lip}(u)$)

$$c\epsilon \leq u(x_0) \leq L\epsilon.$$

We wish to show that there exists $x_1 \in B_\epsilon(x_0)$ such that

$$u(x_1) \geq (1 + \delta)u(x_0),$$

with δ to be specified later.

Assume not, then

$$v(x) := (1 + \delta)u(x_0) - u(x) > 0 \quad \text{in } B_\epsilon(x_0)$$

and solves

$$\mathcal{L}v = -f \quad \text{in } B_\epsilon(x_0).$$

By Harnack inequality,

$$v \leq C(L)(\delta u(x_0) + \epsilon^2 \|f\|_\infty) \quad \text{in } \overline{B}_{c(L)\epsilon}(x_0),$$

with $c(L) = 1 - \frac{c}{4L}$.

Hence, for $\delta < c/4L, \epsilon < c/4\|f\|_\infty$,

$$v \leq C(L)(\delta L\epsilon + \epsilon^2 \|f\|_\infty) \leq \frac{1}{2}c\epsilon \leq \frac{u(x_0)}{2} \quad \text{in } \overline{B}_{c(L)\epsilon}(x_0).$$

From the definition of v it follows that

$$u \geq \frac{c\epsilon}{2} \quad \text{in } B_{c(L)\epsilon}(x_0).$$

However, from the Lipschitz continuity of u it follows that

$$u(x) \leq L(1 - c(L))\epsilon = c\frac{\epsilon}{4} \quad \text{on } \partial B_{c(L)\epsilon}(x_0)$$

a contradiction.

Thus we can construct inductively a sequence of points x_k such that

$$u(x_{k+1}) = (1 + \delta)u(x_k), \quad |x_{k+1} - x_k| \leq Cd(x_k, \partial\Omega).$$

Then using the fact that $\text{dist}(x_k, \partial\Omega) \sim u(x_k)$ and that $u(x_k)$ grows geometrically we find

$$\begin{aligned} |x_{k+1} - x_0| &\leq \sum_{i=0}^k |x_{i+1} - x_i| \leq C \sum_{i=0}^k \text{dist}(x_i, \partial\Omega) \\ &\leq C \sum_{i=0}^k u(x_i) \leq Cu(x_{k+1}) \sim \text{dist}(x_{k+1}, \partial\Omega). \end{aligned}$$

Hence for a sequence of r_k 's of size $u(x_k)$ we have that

$$\sup_{B_{r_k}(x_0)} u \geq cr_k$$

from which we obtain that

$$\sup_{B_r(x_0)} u \geq cr, \quad \text{for all } r \geq |x_0|.$$

The conclusion follows by letting x_0 go to 0.

□

Lemma 5.2. *There exist universal constants $\bar{r}, \bar{C} > 0$, such that*

$$u(x_0) \geq \bar{C} \text{dist}(x_0, F(u)), \quad \text{in } \{x \in \Omega^+(u) : \text{dist}(x, F(u)) \leq \bar{r}\}.$$

Proof. Let $x_0 \in \Omega^+(u)$, $r = \text{dist}(x_0, F(u))$ with $r \leq \bar{r}$ universal to be specified later. Assume first that

$$\text{dist}(x_0, \Omega^+(\underline{u})) > \frac{r}{2}.$$

Thus,

$$(5.2) \quad \underline{u} \leq 0 \quad \text{in } B_{r/2}(x_0).$$

Let $w_k \in \mathcal{F}$ converge uniformly to u , say in $B_R(x_0)$, $r \leq R$. Let us rescale by r around x_0 and use the notation (2.1). Then u_r solves the free boundary problem

$$(5.3) \quad \begin{cases} \mathcal{L}_r u_r = r f_1^+ & \text{in } \Omega_r^+(u_r) \\ \mathcal{L}_r u_r = r f_2^+ \chi_{\{u_r < 0\}} & \text{in } \Omega_r^-(u_r) \\ (u_r)_\nu^+ = G_r((u_r)_\nu^-, x, \nu) & \text{on } F(u_r). \end{cases}$$

Moreover w_k^r converges to u_r uniformly in $B_{R/r}$. Clearly, u_r is the infimum of all admissible supersolutions to (5.3) (in the sense of Definition 1.2) which are above the rescaling $\underline{u}_r(x) = \frac{\underline{u}(x_0 + rx)}{r}$.

We wish to prove that

$$u_r(0) \geq \bar{C},$$

with $\bar{C} > 0$ universal, to be specified later. Assume by contradiction

$$u_r(0) < \bar{C}.$$

By Harnack inequality in $B_1 \subset \Omega_r^+(u_r)$, we have that

$$u_r \leq C(\bar{C} + rM), \quad \text{in } B_{1/2}$$

where $\|f_1\|_{L^\infty} = M$. Hence, for k large enough

$$0 < w_k^r \leq C(\bar{C} + rM) \quad \text{in } B_{1/2}.$$

Now, as in Corollary 4.5, let η, ξ solve the following problems:

$$\begin{aligned} \mathcal{L}_r \eta &= 0 & \text{in } B_{1/2} \setminus \overline{B_{1/4}} \\ \eta &= 0 & \text{on } \partial B_{1/4}, \quad \eta = 1 & \text{on } \partial B_{1/2}, \\ \mathcal{L}_r \xi &= -1 & \text{in } B_{1/2} \setminus \overline{B_{1/4}} \\ \xi &= 0 & \text{on } \partial B_{1/4}, \quad \xi = 0 & \text{on } \partial B_{1/2}. \end{aligned}$$

Call

$$c_0 := \eta_\nu|_{\partial B_{1/4}} > 0, \quad c_1 := \xi_\nu|_{\partial B_{1/4}} > 0$$

with ν the unit normal to $\partial B_{1/4}$ pointing inward and let $c_2 > 0$ be such that

$$(5.4) \quad c_0 c_2 < \frac{G(0)}{2}.$$

Define,

$$v := rM\xi + c_2\eta \quad \text{in } B_{1/2} \setminus \overline{B_{1/4}}.$$

Then,

$$(5.5) \quad \mathcal{L}_r v = -rM \quad \text{in } B_{1/2} \setminus \overline{B_{1/4}}.$$

Moreover, (say $M > 0$) if

$$(5.6) \quad r < \frac{c_2}{2CM} = r_1$$

and \bar{C} is chosen so that

$$(5.7) \quad \bar{C} \leq \frac{c_2}{2C}$$

then,

$$(5.8) \quad 0 < w_k^r \leq \frac{c_2}{2} \leq v \quad \text{on } \partial B_{1/2}.$$

Now, define

$$\bar{w}_r = \begin{cases} w_k^r & \text{in } \Omega_r \setminus B_{1/2} \\ \min\{w_k^r, v\} & \text{in } B_{1/2} \setminus \partial B_{1/4} \\ 0 & \text{in } B_{1/4}. \end{cases}$$

This function is continuous in view of (5.8). Also, from (5.5) and the fact that $w_k^r > 0$ in $B_{1/2}$ it follows that

$$\begin{cases} \mathcal{L}_r \bar{w}_r \leq r f_1^r & \text{in } \Omega_r^+(\bar{w}_r), \\ \mathcal{L}_r \bar{w}_r \leq r f_2^r \chi_{\{\bar{w}_r < 0\}} & \text{in } \Omega_r^-(\bar{w}_r), \end{cases}$$

and from (5.2)

$$\bar{w}_r \geq \underline{u}_r \quad \text{in } \Omega_r.$$

Thus, to assure that \bar{w}_r is an admissible supersolution, we need to require that

$$rMc_1 + c_2c_0 < G(0).$$

In view of (5.4), it is enough to choose

$$r \leq \frac{G(0)}{2Mc_1} = r_2.$$

Thus, for $r \leq \bar{r} := \min\{r_1, r_2\}$ we have reached a contradiction to the minimality of u_r since

$$\bar{w}_r(0) = 0 < u_r(0).$$

□

As a consequence of the two lemmas above, we obtain the following corollary.

Lemma 5.3. *Let $x \in F(u)$ and let A be a connected component of $\Omega^+(u) \cap (B_r(x) \setminus \bar{B}_{r/2}(x))$ such that*

$$\bar{A} \cap \partial B_{r/2}(x) \neq \emptyset, \quad \bar{A} \cap \partial B_r(x) \neq \emptyset,$$

for $r \leq r_0$ universal. Then

$$\sup_A u \geq Cr.$$

Moreover

$$\frac{|A \cap B_r(x)|}{|B_r(x)|} \geq C > 0,$$

where all the constants C depend on $d(x, \partial\Omega)$ and on \underline{u} .

6. THE FUNCTION u IS A SUPERSOLUTION

In this section we prove that u satisfies part (a) in Definition 1.1. First we need to the following preliminary result.

Lemma 6.1. *Let $v_k \geq 0$ satisfy*

$$\mathcal{L}v_k \in L^\infty \quad \text{in } B_2 \cap \{v_k > 0\}.$$

Assume that $v_k \rightarrow v$ uniformly in B_2 . Then

$$\int_{B_1} \frac{|\nabla v_k|^2}{|x|^{n-2}} dx \rightarrow \int_{B_1} \frac{|\nabla v|^2}{|x|^{n-2}} dx.$$

Proof. We sketch the proof. Let V be the fundamental solution of the operator \mathcal{L} . Then $V \sim |x|^{2-n}$ (see [LSW]).

Take a cut-off $\eta \in C_0^\infty(B_2)$, $\eta = 1$ in B_1 . For $w = v$ or $w = v_k$ we have:

$$(6.1) \quad A(x) \nabla v \cdot \nabla w = \frac{1}{2} \mathcal{L}(w^2) - w \mathcal{L}w.$$

On the other hand,

$$\begin{aligned} \int_{B_2} \eta^2 V \mathcal{L}(w^2) dx &= - \int_{B_2} A(x) \nabla(\eta^2 V) \cdot \nabla(w^2) dx \\ &= -2 \int_{B_2} w \nabla w \cdot (A(x) [2\eta V \nabla \eta + \eta^2 \nabla V]) dx \\ &= -4 \int_{B_2 \setminus B_1} w \eta V \nabla w \cdot A(x) \nabla \eta dx - \int_{B_2} A(x) \nabla V \cdot \nabla(w^2 \eta^2) dx \\ &\quad + \int_{B_2 \setminus B_1} w^2 A(x) \nabla V \cdot \nabla(\eta^2) dx \\ &= -4 \int_{B_2 \setminus B_1} w \eta V \nabla w \cdot A(x) \nabla \eta dx - w^2(0) \\ &\quad + \int_{B_2 \setminus B_1} w^2 A(x) \nabla V \cdot \nabla(\eta^2) dx. \end{aligned}$$

Thus we deduce from (6.1) that

$$\int_{B_2} \eta^2 V A(x) \nabla(v_k - v) \cdot \nabla(v_k - v) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

From this the desired result follows using ellipticity and the estimate on V . \square

We also need the following variant of the monotonicity formula in Proposition 3.1 (see again [MP]). We use the same notation as in Proposition 3.1.

Proposition 6.2. *Assume that*

$$u_i(x) \leq \sigma(|x|), \quad x \in B_1, \quad i = 1, 2$$

for a Dini modulus of continuity $\sigma(r)$. Then

$$\Phi(\rho) \leq [1 + \omega(r)]\phi(r) + C\omega(r), \quad 0 < \rho \leq r \leq r_0,$$

with

$$\omega(r) \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

and C depending on $\|u_i\|_{L^2(B_1)}, \sigma, [A]_{0,\gamma}$.

In view of the expansion Lemmas 2.2, 2.3, we only need to prove the next result.

Lemma 6.3. *Let $x_0 \in F(u)$ and*

$$u^+(x) = \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|),$$

and

$$u^- = \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|).$$

Then,

$$\alpha \leq G(\beta, x_0, \nu).$$

Proof. Let $\{w_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$ be a uniformly decreasing to u . As a consequence w_k cannot remain strictly positive in a neighborhood of x_0 , say in a ball $B_r(x_0)$, for all k large. Otherwise u would be a non-negative solution of $\mathcal{L}u = f_1$ in such neighborhood. Then, by standard regularity theory $u \in C^{1,\gamma}$ and $\nabla u(x_0) = 0$. Hence, $u_\nu^+(x_0) = 0$ contradicting the non-degeneracy of u^+ .

For each w_k , let

$$B_{m,k} = B_{\lambda_{m,k}}(x_0 + \frac{1}{m}\nu)$$

be the largest ball centered at $x_0 + \frac{1}{m}\nu$ contained in $\Omega^+(w_k)$, touching $F(w_k)$ at $x_{m,k}$ where $\nu_{m,k}$ is the unit inward normal of $F(w_k)$ at $x_{m,k}$. Then up to proper subsequences we deduce that

$$\lambda_{m,k} \rightarrow \lambda_m, \quad x_{m,k} \rightarrow x_m, \quad \nu_{m,k} \rightarrow \nu_m$$

and $B_{\lambda_m}(x_0 + \frac{1}{m}\nu)$ touching $F(u)$ at x_m , with unit inward normal ν_m . From the behavior of u^+ , we get that

$$|x_m - x_0| = o(\frac{1}{m}),$$

$$\frac{1}{m} + o(\frac{1}{m}) \leq \lambda_m \leq \frac{1}{m}$$

and

$$|\nu_m - \nu| = o(1).$$

Now since $w_k \in \mathcal{F}$, near $x_{m,k}$ in $B_{m,k}$:

$$w_k^+ \leq \alpha_{m,k} \langle x - x_{m,k}, \nu_{m,k} \rangle^+ + o(|x - x_{m,k}|)$$

and in $\Omega \setminus B_{m,k}$

$$w_k^- \geq \beta_{m,k} \langle x - x_{m,k}, \nu_{m,k} \rangle^- + o(|x - x_{m,k}|)$$

with

$$0 \leq \alpha_{m,k} \leq G(\beta_{m,k}, x_{m,k}, \nu_{m,k}),$$

(by Lemma 2.1 the touching occurs at a regular point, for m, k large.) We know that

$$w_k^+ \geq u^+ \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|),$$

hence

$$\underline{\alpha}_m = \liminf_{k \rightarrow \infty} \alpha_{m,k} \geq \alpha - \epsilon_m$$

and $\epsilon_m \rightarrow 0$, as $m \rightarrow \infty$. We have to prove that

$$\underline{\beta} = \liminf_{m,k \rightarrow +\infty} \beta_{m,k} \leq \beta.$$

To do this we argue as follows. If $\underline{\beta} = 0$ there is nothing to prove. Hence let $\beta_{m,k} > 0$. Given r, \bar{x}, v denote by

$$\Phi_r(\bar{x}, v) = r^{-4} \int_{B_r(\bar{x})} \frac{|\nabla v^+|^2}{|x - \bar{x}|^{n-2}} dx \int_{B_r(\bar{x})} \frac{|\nabla v^-|^2}{|x - \bar{x}|^{n-2}} dx.$$

From (3.2) in Theorem 3.5 we obtain that (ρ small)

$$\Phi_\rho(x_{m,k}, w_k) \geq c_n \alpha_{m,k}^2 \beta_{m,k}^2 + o(1),$$

with $o(1) \rightarrow 0$ as $\rho \rightarrow 0$.

Thus, using Proposition 6.2 and letting $\rho \rightarrow 0$, we get that (r small)

$$(6.2) \quad (1 + \omega(r)) \Phi_r(x_{m,k}, w_k) + C\omega(r) \geq c_n \alpha_{m,k}^2 \beta_{m,k}^2.$$

We remark that the w_k^\pm satisfy the assumptions of Proposition 6.2. Indeed the w_k^+ are equiLipschitz. To obtain a uniform modulus of continuity for the w_k^- notice that, the $F(w_k^-)$ have an exterior tangent ball at $x_{m,k}$ of size $1/m$. Thus in a neighborhood of $x_{m,k}$ of size $2/m$ a modulus of continuity independent of k can be obtained building an appropriate barrier. Outside such a neighborhood, the w_k^- inherit the modulus of continuity of the u^- , because w_k^- converges to u^- uniformly.

Now from (3.2), we also have that

$$(6.3) \quad \Phi_r(x_0, u) \geq c_n \alpha^2 \beta^2 + o(1) \quad \text{as } r \rightarrow 0^+.$$

On the other hand, since u^\pm are Lipschitz continuous, for δ small and r small depending on δ

$$\int_{B_r(x_0)} |\nabla u^+|^2 dx = \int_{B_1} |\nabla u_r^+|^2 dx \leq \alpha^2 |B_1 \cap \{x \cdot \nu > \delta\}| + O(\delta) + o(1)$$

Analogously,

$$\int_{B_r(x_0)} |\nabla u^-|^2 dx \leq \beta^2 |B_1 \cap \{x \cdot \nu > \delta\}| + O(\delta) + o(1).$$

By Remark 3.2,

$$\Phi_r(x_0, u) = \Phi_1(0, u_r) \leq \frac{1}{4} \alpha^2 \beta^2 |B_1 \cap \{x \cdot \nu > \delta\}|^2 + O(\delta) + o(1).$$

This, together with (6.3) gives that

$$\lim_{r \rightarrow 0^+} \Phi_r(x_0, u) = c_n \alpha^2 \beta^2.$$

Moreover, since $x_{m,k} \rightarrow x_m$ and $w_k \rightarrow u$ uniformly, we get from Lemma 6.1 that

$$(6.4) \quad \lim_{k \rightarrow \infty} \Phi_r(x_{m,k}, w_k) = \Phi_r(x_m, u).$$

In particular, it follows from that for every $\epsilon > 0$ there exist $r > 0$ small, and m, k large (all depending on ϵ) such that

$$\Phi_r(x_{m,k}, w_k) \leq c_n \alpha^2 \beta^2 + \epsilon.$$

Applying (6.2) and recalling that

$$\liminf_{m,k \rightarrow \infty} \alpha_{m,k} \geq \alpha,$$

it follows that $\underline{\beta} \leq \beta$, because $\alpha > 0$ (by non-degeneracy.) \square

7. THE FUNCTION u IS A SUBSOLUTION

In this section we want to show that u satisfies part b.(ii) in Definition 1.1, that is if $x_0 \in F(u)$ is a regular point from the left with touching ball $B \subset \Omega^-(u)$, then near to x_0

$$u^-(x) = \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \beta \geq 0,$$

in B , and

$$u^+(x) = \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \alpha \geq 0,$$

in $\Omega \setminus B$ with $\alpha \geq G(\beta, x_0, \nu)$.

Notice that, even if $\beta = 0$, then $\Omega^+(u)$ and $\Omega^-(u)$ are tangent to $\{\langle x - x_0, \nu \rangle = 0\}$ at x_0 since u^+ is non-degenerate. Thus u has a full asymptotic development as in the next lemma.

Lemma 7.1. *Assume that near $x_0 \in F(u)$,*

$$u(x) = \alpha \langle x - x_0, \nu \rangle^+ - \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|),$$

with $\alpha > 0$, $\beta \geq 0$. Then

$$\alpha \geq G(\beta, x_0, \nu).$$

Proof. Assume by contradiction that $\alpha < G(\beta, x_0, \nu)$. We will show that in this case we can build a supersolution $w \in \mathcal{F}$ which is strictly smaller than u at some point, contradicting the minimality of u . Let u_0 be the two-plane solution, i.e.

$$u_0(x) := \lim_{r \rightarrow 0} \frac{u(x_0 + rx)}{r} = \alpha \langle x, \nu \rangle^+ - \beta \langle x, \nu \rangle^-.$$

Suppose that $\alpha \leq G(\beta, x_0, \nu) - \delta_0$ with $\delta_0 > 0$. Fix $\zeta = \zeta(\delta_0)$, to be made precise later.

In view of Corollary 4.5, we can find $w_k \searrow u$ uniformly and for r small, k large the rescaling $w_{k,r}$ satisfies the following:

- (i) if $\beta > 0$, then $w_{k,r}(x) \leq u_0 + \zeta \min\{\alpha, \beta\}$ on ∂B_1 ,
- (ii) if $\beta = 0$, then $w_{k,r}(x) \leq u_0 + \alpha \zeta$ on ∂B_1 , and

$$w_{k,r}(x) \leq 0, \quad \text{in } \{\langle x, \nu \rangle < -\zeta\} \cap \overline{B_1}.$$

In particular,

$$w_{k,r}(x) \leq u_0(x + \zeta \nu) \quad \text{on } \partial B_1.$$

If $\beta > 0$, let v satisfy (using the notation in (2.1))

$$(7.1) \quad \begin{cases} \mathcal{L}_r v = r f_1^r, & \{\langle x, \nu \rangle > -\zeta + \epsilon \phi(x)\} \\ \mathcal{L}_r v = r f_2^r, & \{\langle x, \nu \rangle < -\zeta + \epsilon \phi(x)\} \\ v(x) = 0, & \{\langle x, \nu \rangle = -\zeta + \epsilon \phi(x)\} \\ v(x) = u_0(x + \zeta \nu), & \partial B_1 \end{cases}$$

where $\phi \geq 0$ is a cut-off function, $\phi \equiv 0$ outside $B_{1/2}$, $\phi \equiv 1$ inside $B_{1/4}$.

For $\beta = 0$, replace the second equation with $v = 0$.

Along the new free boundary, $F(v) = \{\langle x, \nu \rangle = -\zeta + \epsilon \phi(x)\}$ we have the following estimates:

$$|v_\nu^+ - \alpha| \leq c(\epsilon + \zeta) + Cr, \quad |v_\nu^- - \beta| \leq c(\epsilon + \zeta) + Cr,$$

with c, C universal.

Indeed,

$$v^+ - \alpha \langle x, \nu \rangle^+$$

is solution of

$$\mathcal{L}_r(v - \alpha \langle x, \nu \rangle^+) = g_r \quad g_r := r(f_1^r - \alpha \operatorname{div}(A_r \nu)).$$

Thus, by standard $C^{1,\gamma}$ estimates

$$|v_\nu^+ - \alpha| \leq C(\|v - \alpha \langle x, \nu \rangle^+\|_\infty + [-\gamma + \epsilon \phi]_{1,\gamma} + r\|f_1\|_\infty + r[A]_{0,\gamma}),$$

which gives the desired bound. Similarly, one gets the bound for v_ν^- .

Hence, since $\alpha \leq G(\beta, x_0, \nu(x_0)) - \delta_0$, say for $\varepsilon = 2\zeta$ and ζ, r small depending on δ_0

$$v_\nu^+ < G(v_\nu^-, x_0, \nu),$$

and the function,

$$\bar{w}_k = \begin{cases} \min\{w_k, \lambda v(\frac{x-x_0}{\lambda})\} & \text{in } B_\lambda(x_0), \\ w_k & \text{in } \Omega \setminus B_\lambda(x_0), \end{cases}$$

is still in \mathcal{F} . However, the set

$$\{\langle x, \nu \rangle \leq -\zeta + \epsilon \phi\}$$

contains a neighborhood of the origin, hence rescaling back $x_0 \in \Omega^-(\bar{w}_k)$. We get a contradiction since $x_0 \in F(u)$ and $\Omega^+(u) \subseteq \Omega^+(\bar{w}_k)$. \square

8. THE SIZE OF THE REDUCED BOUNDARY

In this section we prove our regularity Theorem 1.4. First, we need the following standard result.

Theorem 8.1. *Let u be a solution to (1.1), such that u is Lipschitz and non-degenerate. Let $x_0 \in F(u) \cap B_1$ and $0 < \epsilon < \delta < 1$. Then the following quantities are comparable with δ^{n-1} :*

- (i) $\frac{1}{\epsilon} |\{0 < u < \epsilon\} \cap B_\delta(x_0)|$,
- (ii) $\frac{1}{\epsilon} |\mathcal{N}_\epsilon(F(u)) \cap B_\delta(x_0)|$,
- (iii) $N\epsilon^{n-1}$, where N is the number of any family of balls of radius ϵ with finite overlapping covering $F(u) \cap B_\delta(x_0)$.

Proof. We follow the proof of Lemma 10 in [C3]. It suffices to show that

$$(8.1) \quad \int_{\{0 < u < \epsilon\} \cap B_\delta(x_0)} |\nabla u|^2 \sim \epsilon \delta^{n-1},$$

$$(8.2) \quad \int_{B_\epsilon(x_0)} |\nabla u|^2 \sim \epsilon^n.$$

Then, the argument is the same as in the above cited lemma. We notice that the constants in these comparisons depend on the Lipschitz and non-degenerate bounds for u , say C_1, c_1 . In what follows, dependence on C_1, c_1 (as well as the other universal parameters of the problem) is understood and constants depending on these parameters are still called universal.

Inequality (8.2) follows from standard methods (using Poincaré's inequality for the lower bound). Indeed, since u^+ is Lipschitz and non-degenerate

$$\sup_{B_\epsilon(x_0)} u^+ \sim \epsilon, \quad \inf_{B_\epsilon(x_0)} u^+ = 0.$$

To prove (8.1), we rescale

$$u_\delta(x) = \frac{u(x_0 + \delta x)}{\delta}, \quad x \in B_1,$$

and use the notation in (2.1). Let $u_{\varepsilon,s} = \max(s/\delta, \min(u_\delta, \varepsilon/\delta))$, $0 < s < \varepsilon$. Then,

$$\begin{aligned} -\delta \int_{B_1} f_1^\delta u_{\varepsilon,s} &= -\int_{B_1} u_{\varepsilon,s} \mathcal{L}_\delta u_\delta^+ \\ &= \int_{B_1} \langle A_\delta(x) \nabla u_\delta^+, \nabla u_{\varepsilon,s}^+ \rangle dx - \int_{\partial B_1} \langle A(x) \nabla u_\delta^+, \nu \rangle u_{\varepsilon,s} d\mathcal{H}^{n-1} \\ &= \int_{B_1 \cap \{0 < s/\delta < u_\delta < \varepsilon/\delta\}} \langle A_\delta(x) \nabla u_\delta, \nabla u_\delta \rangle dx - \int_{\partial B_1} \langle A_\delta(x) \nabla u_\delta^+, \nu \rangle u_{\varepsilon,s} d\mathcal{H}^{n-1}, \end{aligned}$$

because $\nabla u_{\varepsilon,s} = \nabla u_\delta \cdot \chi_{\{s/\delta < u_\delta < \varepsilon/\delta\}}$.

Hence by ellipticity, using that u^+ is Lipschitz and f_1 is bounded we get ($\delta < 1$)

$$\int_{B_1 \cap \{0 < s/\delta < u_\delta < \varepsilon/\delta\}} |\nabla u_\delta|^2 dx \leq C \frac{\varepsilon}{\delta},$$

with C universal. Letting $s \rightarrow 0$ and rescaling back, we obtain the upper bound in (8.1).

To obtain the lower bound, let V be the solution to

$$(8.3) \quad \begin{cases} \mathcal{L}_\delta V = -\frac{\chi_{B_\sigma}}{|B_\sigma|}, & \text{in } B_1 \\ V = 0, & \text{on } \partial B_1 \end{cases}$$

with σ to be chosen later. By standard estimates, see for example [GT], $V \leq C(\sigma)$ and $-\langle A_\delta \nabla V, \nu \rangle \sim C^*$ on ∂B_1 . By Green formula ($u_\varepsilon = u_{\varepsilon,0}$)

$$(8.4) \quad \int_{B_1} (\mathcal{L}_\delta V) \frac{u_\delta^+ u_\varepsilon}{\varepsilon} - (\mathcal{L}_\delta \frac{u_\delta^+ u_\varepsilon}{\varepsilon}) V = \int_{\partial B_1} \frac{u_\delta^+ u_\varepsilon}{\varepsilon} \langle A_\delta \nabla V, \nu \rangle d\mathcal{H}^{n-1}$$

because $V = 0$ on ∂B_1 . We estimate

$$(8.5) \quad \delta \left| \int_{B_1} (\mathcal{L}_\delta V) \frac{u_\delta^+ u_\varepsilon}{\varepsilon} dx \right| = \left| \int_{B_\sigma} \frac{u_\delta^+ u_\varepsilon}{\varepsilon} dx \right| \leq \bar{C} \sigma,$$

because u is Lipschitz, $0 \leq u_\varepsilon \leq \varepsilon/\delta$. From (8.4) and (8.5) and the fact that $\langle A_\delta \nabla V, \nu \rangle \sim -C^*$ on ∂B_1 we deduce that

$$\begin{aligned} \delta \int_{B_1} (\mathcal{L}_\delta \frac{u_\delta^+ u_\varepsilon}{\varepsilon}) V dx &\geq -\bar{C} \sigma - \delta \int_{\partial B_1} \frac{u_\delta^+ u_\varepsilon}{\varepsilon} \langle A_\delta \nabla V, \nu \rangle d\mathcal{H}^{n-1} \\ &\geq -\bar{C} \sigma + C^* \int_{\partial B_1} \frac{u_\delta^+ u_\varepsilon}{\varepsilon} d\mathcal{H}^{n-1}. \end{aligned}$$

Thus using that u^+ is non-degenerate and choosing σ small enough we get that

$$(8.6) \quad \delta \int_{B_1} (\mathcal{L}_\delta \frac{u_\delta^+ u_\varepsilon}{\varepsilon}) V dx \geq \tilde{C}.$$

On the other hand in $\{0 < u_\delta^+ < \varepsilon/\delta\}$

$$(8.7) \quad \mathcal{L}_\delta \left(\frac{u_\delta^+ u_\varepsilon}{\varepsilon} \right) = \frac{2\delta}{\varepsilon} u_\varepsilon f_1^\delta + \frac{1}{\varepsilon} \langle A_\delta \nabla u_\delta, \nabla u_\delta \rangle.$$

Combining (8.6)-(8.7) and using the ellipticity of A_δ we get that

$$\frac{2\delta^2}{\epsilon} \int_{B_1} u_\epsilon f_1^\delta V + \frac{\delta\Lambda}{\epsilon} \int_{B_1} |\nabla u_\delta|^2 V \geq \bar{C}.$$

From the estimate on V we obtain that for δ small enough

$$\frac{\delta}{\epsilon} \int_{B_1} |\nabla u_\delta|^2 V \geq C$$

for some C universal. Rescaling, we obtain the desired lower bound. \square

Let u be the minimal solution constructed in Theorem 1.3. Then, Theorem 8.1 above implies that $\Omega^+(u) \cap B_r(x), x \in F(u)$ is a set of finite perimeter. Next we show that in fact this perimeter is equivalent to r^{n-1} , and thus conclude the proof of Theorem 1.4. Constant depending possibly on the Lipschitz and non-degeneracy bounds for u are still called universal.

Theorem 8.2. *Let u be the minimal solution in Theorem 1.3. Then, the reduced boundary of $\Omega^+(u)$ has positive density in \mathcal{H}^{n-1} measure at any point of $F(u)$, i.e. for $r < r_0$, r_0 universal*

$$\mathcal{H}^{n-1}(F^*(u) \cap B_r(x)) \geq cr^{n-1},$$

for every $x \in F(u)$.

Proof. The proof follows the lines of Corollary 4 in [C3]. Let $w_k \in \mathcal{F}$, $w_k \searrow u$ in \bar{B}_1 and $\mathcal{L}w_k = f_1$ in $\Omega^+(u)$. Let $x_0 \in F(u)$. As usual, we rescale and use the notation in (2.1):

$$u_r(x) = \frac{u(x_0 + rx)}{r}, \quad w_{k,r} = \frac{w_k(x_0 + rx)}{r} \quad x \in B_1.$$

As in Theorem 8.1, we use the auxiliary function V such that

$$(8.8) \quad \begin{cases} \mathcal{L}_r V = -\frac{\chi_{B_\sigma}}{|B_\sigma|}, & \text{in } B_1 \\ V = 0, & \text{on } \partial B_1. \end{cases}$$

Since $\nabla w_{k,r}$ is a continuous vector field in $\overline{\Omega_r^+(u_r) \cap B_1}$, we can use it to test for perimeter. Denoting for simplicity $w_{k,r} = w$, we get

$$(8.9) \quad \begin{aligned} & \int_{B_1 \cap \Omega_r^+(u_r)} (V \mathcal{L}_r w - w \mathcal{L}_r V) \\ &= \int_{F^*(u_r) \cap B_1} (V \langle A_r \nabla w, \nu \rangle - w \langle A_r \nabla V, \nu \rangle) d\mathcal{H}^{n-1} - \int_{\partial B_1 \cap \Omega_r^+(u_r)} w \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

Using estimates for V and the fact that the w_k are uniformly Lipschitz, we get that

$$(8.10) \quad \left| \int_{F^*(u_r) \cap B_1} V \langle A_r \nabla w, \nu \rangle d\mathcal{H}^{n-1} \right| \leq C(\sigma) \mathcal{H}^{n-1}(F^*(u_r) \cap B_1).$$

As in [C3] we have, as $k \rightarrow \infty$

$$\begin{aligned} & \int_{F^*(u_r) \cap B_1} w \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1} \rightarrow 0, \\ & \int_{\partial B_1 \cap \Omega_r^+(u_r)} w \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1} \rightarrow \int_{\partial B_1} u_r^+ \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1} \end{aligned}$$

and

$$-\int_{B_1 \cap \Omega_r^+(u_r)} w \mathcal{L}_r V \rightarrow \oint_{B_\sigma} u_r^+.$$

Passing to the limit in (8.9) and using all of the above we get

$$(8.11) \quad \left| r \int_{B_1 \cap \Omega^+(u_r)} V f_1^r + \oint_{B_\sigma} u_r^+ + \int_{\partial B_1} u_r^+ \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1} \right| \\ \leq C(\sigma) \mathcal{H}^{n-1}(F^*(u_r) \cap B_1).$$

Since u is Lipschitz and non-degenerate, for σ small

$$\oint_{B_\sigma} u_r^+ \leq \bar{C} \sigma$$

and using the estimate for $\langle A_r \nabla V, \nu \rangle$

$$-\int_{\partial B_1} u_r^+ \langle A_r \nabla V, \nu \rangle d\mathcal{H}^{n-1} \geq \bar{c} > 0.$$

Also, since f_1^r is bounded

$$\int_{B_1 \cap \Omega_r^+(u_r)} V f_1^r \leq \bar{C}(\sigma).$$

Hence choosing first σ and then r sufficiently small we get that the left-hand-side in equation (8.11) is larger than a constant \tilde{C} , which concludes our proof. \square

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